From the Hole Argument (A.Einstein) to the Ball of Clay Argument (H. Weyl)

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C5. Physics,

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<u>Short Abstract :</u>

The *hole argument* is a story invented by Albert Einstein to set out the difficulties he had in reconciling his principle of covariance with the Mach's principle. Some modern presentations insist that the bare differential manifold loses its physical status and is just an expression of the mathematical framework of the new relativistic theory.

In *Space-Time-Matter*, Hermann Weyl constructed an argument close to the modern hole argument, replacing the hole with a ball of clay. Beside some trivial technical differences, Weyl's argument is original because of the particular concept of mathematics that he inherited from the Gottingen school.

Les signes (**********) représentent les moments où il faut changer de diapositive Powerpoint)

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I will talk first about the

<u>Aim of this paper:</u>

In 1918, Hermann Weyl published *Space-Time-Matter,* one of the first general handbooks on General relativity. Weyl thought with enthusiasm that *Albert Einstein's theory is a brilliant confirmation of Bernhard Riemann's epistemological ideas on geometry*. *That involves a redistribution of the roles of mathematics and physics in the scientific constitution of the space concept.* In a short text from *Space-Time-Matter*, Weyl explains how this redistribution works in a theory like general relativity, that is in a Riemannian space where the metrical coefficients are determined by matter. For his explanation, he uses an argument very close formally to the famous Albert Einstein's Hole argument. But, in Weyl's argument, the hole is replaced by a ball of clay which is modeled by hand. We will call it the "ball of clay argument".

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We will present two different points of view on the BCA.

Firstly, we will provide a comparison between Einstein's Hole Argument and Weyl's BCA. This comparative interpretation will show some differences in the mathematical and physical content of the arguments. And that will highlight some features of GTR-like theories.

Secondly, we will propose an intrinsic interpretation of the BCA. That is a reading from the point of view of Weyl himself. We will see that the excessive simplifications that we meet in Weyl's expression of Mach's principle can be explained by a difference in the philosophical problems guiding both authors. Beyond its formal analogy with the HA, the aim of the BCA is different. It presents an important dilemma, which could be the key idea in understanding the unity of *STM*, and therefore, the unity of Weyl's thoughts on the space concept.

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I start by presenting quickly

<u>Einstein's presentation and the modern presentation of the *hole argument* :</u>

Albert Einstein struggled with both mathematical and conceptual difficulties during the years 1907-1915, when he was searching for a relativistic theory for gravitation. Einstein wanted his field theory to satisfy two strong hypotheses, namely 1) the total determination of the inertia by the distribution of matter (Mach's principle) and 2) the principle of general relativity. He thought that the latter had to be expressed by a general covariant theory, in the context of a Riemannian metric. There $fore, he was looking for tensorial field equations $G_{ab}(x) = T_{ab}(x)$, where $T_{ab}(x)$ is the stress-energy tensor (representing mass), and $G_{ab}(x)$ is a mathematical function of the metrical tensor $g_{ab}(x)$ and its derivatives. However, for a while, some of his difficulties appeared to suggest that such a general schema couldn't provide a correct expression of Mach's principle.

The *hole argument*, developed by Einstein, was a didactic way to present his momentary belief in this incompatibility.

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What Einstein wanted to show, in his hole argument, is that, in every theory of the previous kind, there is a kind of indeterminism, in the sense of a violation of Mach's principle.

In such a theory, we could find, *for the same field of stress-energy coefficients* $T_{ab}(x)$, two different fields of metrical coefficients $g_{ab}(x)$ and $g'_{ab}(x)$ that both verify the gravitational equations (*). The second field is constructed from the first, simply using the fact that the gravitational equations (*) are covariant, and using the hypothesis of a region of space (*the hole*) that is void of matter. Einstein concluded first that, if we took these kinds of covariant gravitational equations, we had to introduce into the physical theory some indeterminism–something which was unacceptable for him.

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Let's focus now on a modern presentation of the argument (we think about the interpretation of three Johns: J.Stachel, J.Earman and J.Norton). In their interpretation of the hole argument, they minimize the importance of Mach's principle, and so they minimize the importance of the hole itself.

For them, what is important to understand is that, before we put any field on, space-time as a purely topological manifold is not already a physical entity. It is just a mathematical frame.

We just have to understand that two solutions differing only by an active diffeomorphism are two *mathematical* expressions of the same *physical situation*.

We see that the issue, here, differ from that of the original argument.

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Three notions compose the hole argument: the manifold, the metric and the mass (the stress-energy tensor). The modern presentation insists on the relation between on one side, the bare manifold, and, on the other side, the metrical field and the stress-energy field together. Thus, we focus here on *the epistemological problem about the relation between the mathematical framework and the physical entities.*

In fact, what Einstein really wanted to compare was, on one hand, the manifold with the stress-energy tensor defined on it, and on the other hand, the metrical tensor. The main problem therefore was to think the relation between two physical realities (mass and metric), an ontological problem about the nature of the relations between the metric and the masses.

To solve this original problem, we have to understand that we still have a mathematical freedom in the way we express the metric on the manifold, even when the values of the stress-energy tensor are given everywhere.

Let's turn now to the ball of clay argument.

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Contrary to Einstein and the moderns, Weyl uses excessively simplified hypotheses to develop his argument:

1) He posits that the (differential) space-time manifold is R^4 itself
2) He models the content of space-time (matter and energy) by a single scalar function ρ.
3) He gives Mach's principle in a very simple form. He states that the metric is directly a function of the repartition of matter.

Like in the hole argument, Weyl's argument is developed in three steps. (we change just a little the order inside the argument) We start from a certain repartition of matter $\rho(x)$ with a metric $g_{\mu\nu}(x)$. Then, we change the repartition of matter to $\rho'(x)$. Finally, we change the frame of coordinates in such a way that the new repartition of matter is expressed in the new frame of coordinates, like the old repartition of matter was expressed in the first frame of coordinates. (Then, have ρ at step 1 and then again ρ at step 3, the same mathematical function).

Now, we explore two possibilities for the behavior of the metric.

Possibility a)

If the metric is independent from matter. Then, at step 2, the metric does not change. But, when we change the frame of coordinates, at step 3, we change the coefficients of the metric. Therefore, they differ at the end from the initial coefficients. Weyl interpret this by saying that matter cannot move from one point of Space to another without changing its metrical relations.

Possibility b)

If the metric is dependant from matter. Then, the metric changes at step 2. And when change the frame of coordinates at step 3, then the coefficients come back to their first values. We have the same mathematical functions $g_{\mu\nu}$ at the end as at the beginning. Weyl interpret this by saying that matter can move from one point of space to another, without changing its metrical relations. Space is than homogeneous. We have, in one sense, the possibility of a rigid motion.

What is important for Weyl is to show that, even in a Riemannian context, we can think about the possibility of a rigid motion, or of the homogeneity of space. In other words, he wants to show that we can move a body from a point of space to another, without changing its intrinsic metrical relations.

Therefore, Weyl concludes that only the case b) is correct. *We have to suppose that the metric is dependant from matter*, if we want to save homogeneity of space in a Riemannian context.

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Then, Weyl comes to what we can call properly the "Ball of Clay Paradox". He states that:

> "The simple fact that I can squeeze a ball of modeling clay with my hands into any regular shape totally different from a sphere would seem or reduce Riemann's view to absurdity."

Space-Time-Matter, §12

Indeed, Weyl wanted to illustrate the *possibility* of a rigid motion in a Riemannian context. But the conclusion of his argument is not the *possibility* but the *necessity* of the rigidity of motion. The universe as a whole appears in his argument as a unique rigid body where any motion is impossible, even the simple possibility to change the shape of a ball of clay.

Thus, Weyl's ball of clay paradox is a paradox precisely in the same sense as Zeno's paradoxes. It demonstrates the impossibility of any motion, starting just from a few rational hypothesis.

A new kind of paradox of motion like Zeno's

1) *If the inner properties of matter can be entirely determined by fields whose signification need not any metric (like scalar fields*

2) *And if these inner non-metrical properties of matter entirely determine the metric*

Then no motion is possible is the universe (without changing the inner properties of matter).

In the text following our quotation, Weyl tries several ways to solve the paradox. Nevertheless, we think that these solutions are less interesting than the reason why a paradox of motion is implied in Weyl's ball of clay argument, but not in Einstein's hole argument.

Indeed, we have a paradox of motion in Weyl's schema, because of his over simplified formulation of Mach's principle. So, by contrast with Einstein's more subtle schema, it demonstrates that, in order to be compatible with the possibility of motion:

1) Mach's principle has to be expressed in a Local-Dynamic (not a Global-Static) form. We have to take time into account in order to express correctly Mach's principle

2) Mach's principle cannot link a totally non-spatial concept of matter with the metric. Rather, it must link matter, *as it is distributed in a current metrical web,* with the future distribution of matter. Matter does not determine the metric but the *evolution* of the metric.

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Now, we turn to the intrinsic approach to the Ball of Clay argument.

We saw that the Ball of Clay Argument of Weyl has mathematical and physical weaknesses with regards to the hole argument of Einstein, because of excessive simplifications. But, beside this, the difference between these arguments highlights an important change in the philosophical issues of these two authors.

Weyl simplifies the modeling of matter because, like in modern interpretations of the hole argument, his main interest is not the way in which matter can entirely determine the metric. For Weyl, this determination is no longer a *problem* but an *hypothesis*.

Instead of this, Weyl's problem is now:

> *In the context of a Riemannian metrics whose coefficients are determined by matter, can we continue to think about geometry as an a priori mathematical knowledge about a space characterized by its homogeneity?*

We see that Hermann Weyl's problem is closer to the modern approach of the hole argument than to Einstein's. But his investigation into the link between mathematics and physics in geometry is greatly influenced by a particular concept of mathematics, which is inherited from the Gottingen school.

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To save the homogeneity of space in this Riemannian context, in the text of the Ball of Clay Argument, he adopts a radical solution. He excludes the notion of metric from the domain of the mathematical geometry, in the strict sense. The space, as the homogeneous object of geometry, is reduced to the topological (differential) notion of manifold. This idea is closely linked to some idea we can find in several Einstein's texts. We have to wait for the third edition of *Space-Time-Matter* to read about a more subtle solution to Weyl's problem. This more elaborate solution is supplied by the idea of "Nahegeometrie" (we refer to the works of Erhard Scholz for this notion) The idea is that only the infinitesimally closed relations are the object of an *a priori* knowledge, whereas relations at a finite distance need the consideration of matter and so is the object of physics.

To conclude, we have first highlighted some important features of Mach's principle, by a comparative interpretation of the Ball of Clay argument with the hole argument. The, we turned to an intrinsic interpretation of the text of the Ball of Clay Argument. And we saw that it can be considered as the place, in *Space-Time-Matter*, where Hermann Weyl puts down the problem of the conciliation of Riemannian geometry with the idea of the homogeneity of space. That idea of homogeneity was very important for Hermann Weyl, because he defended some kind of idealism of space. Therefore, the issue presented in the Ball of Clay Argument is the key to have a global understanding of the great geometrical works of this author.

This is the birth of the epistemological problem that has guided Hermann Weyl to his Nahegeometrie.

Thank you.